Self-organized criticality: Robustness of scaling exponents

Jozef Černák*
University of P. J. Šafárik, Department of Biophysics, Jesenná 5, SK-04000 Košice, Slovak Republic
(Received 17 November 2001; published 10 April 2002)

We investigate a deterministic, conservative, undirected, critical height sandpile model with dissipation of an energy at boundaries that can simulate avalanche dynamics. It was derived from the Bak-Tang-Wiesenfeld model [P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987)] introducing an additional second-higher threshold so the model has two distinct thresholds. Our computer simulations for a two-dimensional lattice show that scaling properties of the model depend on the higher-threshold values and site concentrations. These results are not therefore consistent with the present self-organized criticality hypothesis where the scaling properties are independent of the model parameters.

DOI: 10.1103/PhysRevE.65.046141 PACS number(s): 05.70.Jk, 05.40.−a, 05.70.Ln

I. INTRODUCTION

Bak, Tang, and Wiesenfeld (BTW) developed a simple model to describe the behavior of spatially extended dynamical systems—systems with both temporal and spatial degrees of freedom [1,2]. They introduced a theoretical framework, the so-called self-organized criticality (SOC) as a common underlaying mechanism of ubiquitous "$1/f$" noise, self-similar fractal structures, and turbulence. The SOC idea stimulated numerous experimental [3–11], numerical [12–24], and theoretical [25–28] studies.

The sandpile was introduced as an example to illustrate the basic idea of SOC in real systems [2]. The first tests of the SOC model in granular materials have not confirmed the direct analogy between the dynamics of sandpiles and physical systems exhibiting SOC [3,4]. These results led to the conclusion [5] that granular materials do not show SOC and it was proposed that first-order behavior may be the generic result and SOC might be the exceptional situation. Frette *et al.* [6] demonstrated experimentally the crossover from critical to noncritical behavior in a pile of rice. In one case, for grains with a large aspect ratio the dynamics exhibits SOC, but not in another case for less elongated grains. They concluded that SOC is not as "universal" as was initially assumed [1,2], and the detailed mechanisms of energy dissipation plays an important role. Altshuler *et al.* [8] observed that the nature of the sandpile base is a parameter which can modify the avalanche dynamics in slowly driven one-dimensional piles of beads. They also discussed the role of quenched and unquenched randomness of the avalanche dynamics.

The notion of sandpiles were not the only model system to study SOC. There are also many other real systems that exhibit avalanche behavior, for example avalanchelike phenomena in magnetic materials [9,29], and a simple stick-slip process of dragging a sandpaper across a carpet [10]. In this case the BTW model was extended to nonconservative systems by replacing the updating rule so that the energy dissipation was occurring at any time scale. In a superconductor, a distribution of vortex avalanche sizes showed the power-law behavior over two decades, proving that the vortex dynamics in the Bean state is characterized by avalanches of many length scales [11]. In these experiments the critical exponents from avalanche size distributions were not constant but they depended on the biased external magnetic fields.

Kadanoff *et al.* [12] numerically investigated scaling and multiscaling behavior of one- and two-dimensional SOC models. They used different microscopic updating criteria to see their influence on the scaling and multiscaling properties. They observed the several universality classes and different models with similar rules belong to the same class. All previous numerical models were defined with deterministic update criteria although Manna [13] changed the microscopic dynamics using a stochastic relaxation rule that enables toppling of the sand in the randomly selected directions.

The nonconservative SOC model was introduced by Olami, Feder, and Chistensen (OFC) [15]. Their model displayed robust scaling behavior for different strength of dissipation. On the other hand, the scaling exponent was nonuniversal and depended on the elastic parameter. This model was thus used for a long time for systems that showed avalanche dynamics described by a power law, but with a nonuniversal scaling exponent [11,23]. However, recent results for larger lattices [24] than in the OFC paper [15] confirm universality of the scaling exponent in a wide range of the model parameter. This exponent agrees with the exponent of the GR model of earthquakes. Jánosi and Kertész [16] influenced by the OFC model investigated its properties in a framework of a sandpile. They observed a difference between the dissipative and conservative models when a quenched randomness in the threshold values was introduced. They concluded that quenched randomness in dissipation models can destroy SOC, but if conservation is present, then disorder is irrelevant. Head and Rodgers [19], inspired by the experiments of Frette *et al.* [6], have developed a model which incorporates both individual particle and cluster motions. This model exhibits avalanche distributions described with stretched exponential or power-law dependence that agree with the rice-pile experiments [6].

Zhang [25] introduced a variant of the BTW model where an energy is continuous. All the energy from an unstable site is distributed to the neighbors and after relaxation the site

*Electronic address: jcernak@kosice.upjs.sk

remains empty. Dhar [26] generalized the BTW model introducing a set of thresholds that are assigned to each site. He proved that the critical height BTW model is a subclass of the mathematical structure, *Abelian* group, and named it *Abelian* sandpile model (ASM). If the configuration after the avalanches does not depend on the order in which the relaxations of the active sites were performed, then the model belongs to the ASM. Later [27], Dhar proved the *Abelian* properties of the stochastic Manna's model. Based on the renormalization-group transformation study, Pietronero *et al.* [28] concluded that both the deterministic BTW [1] and the stochastic Manna's model [13] belong to the same univesality class and microscopic details of an energy relaxation have no effect on the scaling properties. This conclusion started a broad discussion about a classification of deterministic and stochastic ASM as well as all known sandpile systems [18,20,22,23,30].

The various models we have reviewed here thus show support or limited support for the SOC hypothesis. Among there, a few papers motivated us to test the robustness of scaling properties (i) the conclusion about robustness of the BTW model [2] and (ii) the rice-pile experiment [6,7].

The paper is organized as follows. In Sec. II, the sandpile model is defined. The results describing the avalanche area distributions such as scaling exponents, and average avalanche areas at different parameters are presented in Sec. III. This section also contains a small part concerned with the noise properties of avalanche dynamics. We discuss our results with simulations made previously in Sec. IV. In the last section in Sec. V, we summarize our results and suggest various ways to explore them further.

II. DEFINITION OF MODEL

We follow a notation presented by Biham *et al.* [23] to define a sandpile model. Consider a d-dimensional hypercubic lattice of linear size L. Each site $\mathbf{i}$ has assigned a dynamical variable $E(\mathbf{i})$ that generally represents a same physical quantity such as energy, grain density, stress, etc. A configuration $\{E(\mathbf{i})\}$ classified as stable if for all the sites $E(\mathbf{i}) < E_c$, where E_c is a threshold value. In our model the threshold value E_c is not constant, but depends on the site position $\mathbf{i}$, $E_c(\mathbf{i})$. As consequence of this we have to generalize the condition for the stable configuration $\{E(\mathbf{i})\}$ and is now: $E(\mathbf{i}) < E_c(\mathbf{i})$. There are many ways to define $E_c(\mathbf{i})$, but in our case $E_c(\mathbf{i})$ has only two distinct values

$$E_c(\mathbf{i}) = \begin{cases} E_c^A = 2d \\ E_c^B = 2dk, \quad k = 2,3,4,\ldots, \end{cases} \tag{1}$$

where d is a dimension and k is an integer number. For any site $\mathbf{i}$ we define the threshold $E_c(\mathbf{i})$ [Eq. (1)] in such a manner that n randomly chosen sites have the higher value E_c^B and the remaining $L^d - n$ sites have the lower value E_c^A. The concentration of sites with higher-threshold value E_c^B is denoted c, and $c = 100n/L^d$ [%].

Let us assume that a stable configuration $\{E(\mathbf{j})\}$ is given, and that we select a site $\mathbf{i}$ at random and increase $E(\mathbf{i})$ by some amount δE. When an unstable configuration is reached, $E(\mathbf{i}) \geqslant E_c(\mathbf{i})$, and a relaxation takes place. An unstable site $\mathbf{i}$ lowers energy, that this is distributed among the neighbor sites. This event is described by the relaxation rules

$$E(\mathbf{i}) \rightarrow E(\mathbf{i}) - \sum_e \Delta E(\mathbf{e}), \tag{2}$$

$$E(\mathbf{i}+\mathbf{e}) \rightarrow E(\mathbf{i}+\mathbf{e}) + \Delta E(\mathbf{e}), \tag{3}$$

$$\sum_e \Delta E(\mathbf{e}) = E_c(\mathbf{i}), \tag{4}$$

where $\mathbf{e}$ is a set of vectors from the site $\mathbf{i}$ to its neighbors. The neighbors that receive an energy can become unstable and topple thus generating an avalanche. To the two well-known rules expressed in Eqs. (2) and (3) that describe a distribution of an energy, we added an additional rule Eq. (4) that specifies the manner how an energy is distributed depending on the site position $\mathbf{i}$, and corresponding threshold $E_c(\mathbf{i})$. This rule tells us that an energy equal to the discrete amount of $E_c(\mathbf{i})$ is transferred from an unstable site to the neighbors. Specifically, in our model there are two distinct amounts of transferred energy Eq. (1) to be considered. The relaxation rules Eqs. (2)–(4) are applied until that moment when a stable configuration is reached again, for all sites $\mathbf{i}$, $E(\mathbf{i}) < E_c(\mathbf{i})$. Obviously, during one avalanche an arbitrarily unstable site $\mathbf{i}$ can transfer the energy $E_c(\mathbf{i})$ a few times to be stable, $E(\mathbf{i}) < E_c(\mathbf{i})$. A d-dimensional lattice with open boundaries has been investigated so an added energy can flow outside the system, and an energy dissipation takes places only at boundaries.

This model has been designed to have the specific properties that could simplify it. It belongs to the critical height models [12,14] with conservative relaxation rules and an undirected energy transfer (particle sliding) [12]. We characterize the model as deterministic with a frozen randomness of the thresholds $E_c(\mathbf{i})$ [16]. This means that no quantity is randomly perturbed as in other models [7,31,32].

III. RESULTS

We shall here report the results obtained using computer simulations of a deterministic, conservative, undirected, and critical height sandpile [12,14] model defined by Eqs. (1)–(4). The simulations were carried out for the following parameters: $d=2$, two-dimensional lattice of linear sizes $L = 256$ and 512, randomly added energy $\delta E = 1$, lower-threshold $E_c^A = 4$, higher threshold $E_c^B = 8$, 16, 32, 64, 128, and 252, and with concentrations of sites with the higher threshold values in the range $c = 0.05 - 50\%$. The second threshold E_c^B and concentration c, are also considered as model parameters. The model is *non-Abelian* (Sec. IV) and we therefore allow toppling only at one randomly selected site. Avalanches can be characterized by such properties as their size, area, and lifetime [20,22,23]. We measure only one property, the avalanche area a that is the number of lattice sites that have relaxed at least one during the avalanche [1,23].

The simulations in the two-dimensional lattice of the lin-

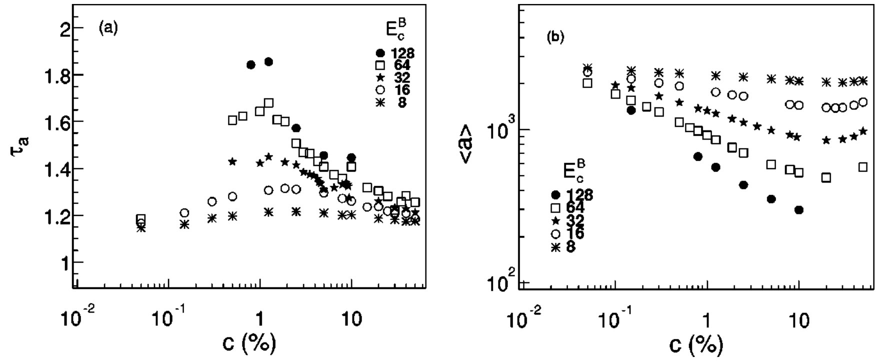

FIG. 1. Computer simulations carried out on a two-dimensional lattice of the linear size $L=256$, with the lower threshold $E_c^A=4$. (a) The exponent τ_a versus concentration for various thresholds $E_C^B=8(*)$, 16(○), 32(★), 64(□), and 128(●). (b) The average area of avalanches $\langle a\rangle$ versus the concentration c for different thresholds E_c^B. These curves show minima for certain parameters c and E_c^B.

ear size $L=256$ show that the avalanches have areas a that can span a wide range of sizes up to the whole system. The distribution of avalanche areas $P(a)$ obeys the power-law $P(a)\propto a^{-\tau_a}$, as it was expected. For the model parameters mentioned above we have determined the critical exponents τ_a, and they are shown in Fig. 1(a). We can see that for small value of the higher threshold E_c^B, $4<E_c^B\leqslant 16$, the exponents weakly depend on the concentration c. When the higher values of E_c^B, $E_c^B\geqslant 32$ are considered, the exponents reach maxima at concentration $c\approx 1\%$. We have observed, that for a concentration $0.05<c<1\%$, the curves deviated from the expected power-law dependence due to the fact that it was not possible to fit the results to a power law. The corresponding values of exponents are thus missing in this graph. We can therefore conclude that in this range of concentration c, the avalanche area $P(a)$ does not show scaling properties.

The average avalanche area $\langle a\rangle$ is sensitive to the choice of parameters, as it is presented in Fig. 1(b). By increasing the higher threshold values E_c^B at a constant concentration c, one gets decreasing avalanche areas $\langle a\rangle$. When concentration c is changed at constant E_c^B, $E_c^B\geqslant 16$ we have observed decreasing the avalanche areas $\langle a\rangle$ for concentrations $0.05<c<10\%$ down to a minimum, and subsequent increasing values when the concentration increases. In the concentration range $0.05<c<10\%$, the equation $\langle a\rangle\propto c^{-\alpha}$ is valid, and where $\alpha=0.177$ ($E_c^B=32$), 0.271 ($E_c^B=64$), and 0.388 ($E_c^B=128$). By tuning these parameters, we can obtain a state where the average avalanche areas $\langle a\rangle$ reach minima.

We have also simulated the model in a larger two-dimensional lattice of linear size $L=512$ to eliminate the effect of finite sample size on the characteristic exponents τ_a. The results are presented in the Table I. The characteristic exponents τ_a are smaller than in the previous case when $L=256$, but it is reasonable to assume that they are closer to the exact values. The dependence of the exponents τ_a on the concentration c at the higher threshold $E_c^B=128$ shows the maximum exponent $\tau_a=1.451$ at a concentration $c=1\%$. When the concentration c increases to $c=2\%$, the exponent decreases to $\tau_a=1.279$, similarly as in the Fig. 1(a). The results concerning to the average avalanche area $\langle a\rangle$ obey the power law $\langle a\rangle\propto c^{-\alpha}$ with $\alpha=0.272$ (fit is valid for $c=1-8\%$) and are consistent with results presented in Fig. 1(b). Comparisons of two avalanche area distributions $P(a)$ for two distinct concentrations $c=1$ and 8% are presented in Fig. 2. This shows that there is an essential difference for small avalanche areas $a<1.8\times 10^3$. For the concentration $c=1\%$ and avalanches in the interval $a<1.8\times 10^3$, the avalanche area distribution cannot be fitted to a power law. For larger avalanches $1.8\times 10^3<a<2\times 10^5$, the avalanche area distribution obeys the power law with $\tau=1.451$. For a concentration $c=8\%$, we see that at $a\approx 200$ the slope of the curve changes. A shift in the ratio between small and big avalanches towards smaller avalanches ($a<200$) was observed. For larger avalanches $a>2\times 10^3$, the avalanche area distribution is approximated by the power law with the exponent $\tau=1.279$. Both avalanche area distributions thus scale for larger avalanches, but the scaling exponents are not universal and depend on the parameters.

We have analyzed the time Δt between two avalanches [33] and the power spectrum of a signal $X(t)$ generated by the avalanche dynamics with an explicitly defined time scale [32]. The time signal is given as

$$X(t)=\sum_j a_j\delta(t-t_j). \tag{5}$$

Here, a_j denotes the area of jth avalanches and t_j is the time of its occurrence. We denote the time between two ava-

TABLE I. Critical exponents τ_a, average avalanche areas $\langle s\rangle$, and characteristic times ΔT_0 for different thresholds E_c^B, and concentrations c. The sandpile model is two-dimensional, conservative, undirected, and deterministic of a linear size $L=512$. The avalanches which areas are from the interval $a=1.8\times 10^3-2\times 10^5$ are considered to fit the critical exponent τ_a.

E_c^B	c	τ_a	$\langle s\rangle$	ΔT_0
128	1	1.451	1404	1.733
128	2	1.373	1135	1.797
128	4	1.318	937	1.919
128	8	1.279	808	2.153
252	10	1.279	419	2.289

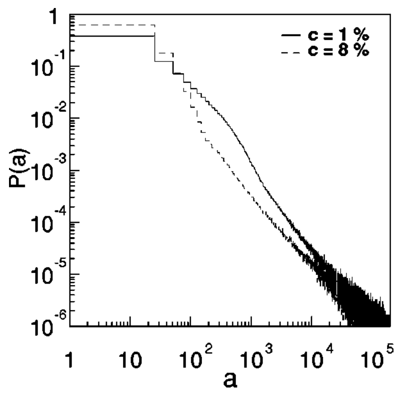

FIG. 2. Comparison of the avalanche area distributions $P(a)$ for two different concentrations $c=1\%$ and $c=8\%$, where the remaining parameters are: the lower threshold $E_c^A=4$, higher threshold $E_c^B=128$, linear lattice size $L=512$. Corresponding exponents are determined: $\tau_a=1.451\pm0.003$ ($c=1\%$), and $\tau_a=1.279\pm0.003$ ($c=8\%$) for avalanches from the interval $a=1.8\times10^3-2\times10^5$.

lanches as $\Delta t=t_j-t_{j-1}$. The original time signal $X(t)$ is transformed using the Fourier transform into the frequency domain. The square of the amplitudes of the transformed signal at a given frequency range define the power spectrum

$$S(f)=\lim_{T\to\infty}\frac{1}{2T}\left|\int_{-T}^{T}dtX(t)e^{-i2\pi f}\right|^2. \tag{6}$$

The resulting power spectrum is shown in Fig. 3 for the parameters $L=256$, $E_c^B=128$, and $c=1\%$ with no evidence of $1/f$ noise. We see a short-frequency interval where $S(f)\sim1/f^2$. The fast Fourier transformation was computed from $N=524\,288$ samples [34].

We have observed that the distributions $P(\Delta t)$ of the intervals Δt between avalanches exponentially decays as $P(\Delta t)\propto\exp(-\Delta t/\Delta T_0)$. The characteristic interval ΔT_0 is sensitive to the change of a concentration c and higher threshold E_c^B, as is shown in Table I.

It was observed that the area distribution consists of three main parts: (i) region of small avalanches, (ii) regions of intermediate, and (iii) regions of larger avalanches that span whole sample. The boundary between small and intermediate avalanches depends on the model parameters. The scaling exponent τ_a determined in the intermediate region is not universal. The model does not show any evidence of $1/f$ noise.

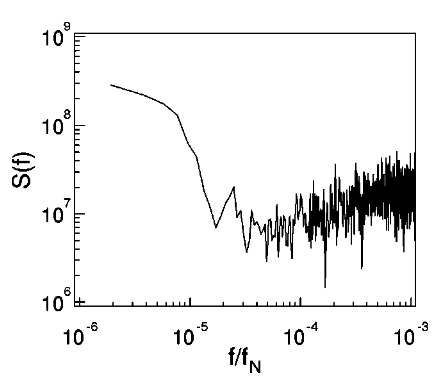

FIG. 3. The power spectrum $S(f)$ for a time signal $X(t)$ defined by Eq. (5), and f_N is Nyquist frequency. The avalanche dynamics was generated by parameters: $L=256$, $E_c^B=128$, and $c=1\%$.

IV. DISCUSSION

The critical point in SOC systems is an attractor which is reached by starting far from equilibrium, and the scaling properties of the attractor are insensitive to the parameters of the model. To reach this state no tuning of parameters is necessary from the outside, i.e., the system organizes itself [1,2].

Bak *et al.* [2] tested the robustness of scaling exponent of the BTW model by removing as many as 25% of the lattice bonds, but the avalanche dynamics still led to a power-law distribution. By removing about 10% of the bounds, still no change in the exponent was detected. This test confirmed the SOC hypothesis. We chose another type of defect in order to increase the disorder in the BTW model. In our model there is one of two distinct thresholds [Eq. (1)] assigned to each site. Lattice sites **i** with a higher threshold are considered for lattice defects. The model defined in such a manner has a short-range spatial correlation introduced into the update rules. This means that the toppling of a defective site can influence more than $2d$ sites as in the BTW model at one relaxation. An unstable site also makes its surroundings unstable and not only neighbors and a collective toppling starts. A correlation length is proportional to the ratio E_c^B/E_c^A. The defect behaves as a local energy reservoir. It can absorb and discharge more energy than its neighbors, so the avalanche dynamics near defects is changed. We can observe the different nature of the defects introduced before [2] and those presented here. By removing bounds in a lattice, this causes a permanent barrier to avalanche toppling and some parts of a system remain inactive. On the other hand, a site which behaves as an energy reservoir enables avalanche spreading and can introduce a time delay (an accumulation of energy) and short range correlations (a relaxation) into the dynamics of the system. Our model displays unexpected properties (Sec. III), in that the avalanche dynamics depends on the choice of parameters. This finding disagrees with the SOC hypothesis [1,2].

The model is a special case of the ASM [26,27] when the elements of the integer matrix Δ are defined as $\Delta_{ii}=E_c(\mathbf{i})$ from Eq. (1), but, it is not *Abelian* for all model parameters. Let us consider two nearest-neighbor sites 1 and 2 that are active simultaneously, $E_c(1)=8$, $E(1)=7$, $E_c(2)=4$, $E(2)=3$, $E(i)=0$ for all $i>2$, and $\delta E=1$. If site 1 topples before 2, in the resulting configuration $E(1)=1$ and $E(2)=1$. If the order of toppling is reversed, we obtain $E(1)=0$ and $E(2)=2$, and the resulting configuration depends on the toppling order and thus the model is *non-Abelian*. If we choose the special parameters $\delta E=k$, and $E_c(\mathbf{i})=2dk$ from the Eq. (1) then the model is *Abelian*.

Manna [14] observed the dependence of the exponents τ on the lattice size L in systems with finite-size behavior and described this with the relation

$$\tau(L)=\tau_\infty-\text{const.}/\ln(L). \tag{7}$$

This property has been explored by Lübeck and Usadel [35] to determine τ_∞ for the BTW model. If we compare our exponents for the different lattice sizes $L=256$ [Fig. 1(a)], and $L=512$ (Table I) we can see an opposite tendency in τ_∞ when the lattice size is increased. We explain this by an occurrence of the critical avalanche area (Fig. 2) that divides the dependence into two main regions. First, where small avalanches dominate, a deviation from a power law is clearly evident for some parameters, and a second region, a power-law approximation, is valid for larger avalanches. The critical avalanche area was not so clearly evident if $L=256$ and could probably shift the fits to higher values. Despite this fact we assume that the exponents at $L=512$ are close to the asymptotic values if $L\rightarrow\infty$ and subsequently increasing L one should approach the right values. Note that our exponents for the avalanche area at $L=512$ are equivalent or higher than the asymptotic value $\tau_{d,\infty}\approx 1.33$ [35]. In our case we do not know anything about $\tau(L)$ dependence when $L>512$. Will $\tau(L)$ dependence follow Manna's observation given by Eq. (7)? Then the exponents determined here could be higher. We should however be careful because Manna's finding is not confirmed by any theory [18].

For the second threshold $E_c^B \geqslant 16$, we have observed a decreasing tendency of the average avalanche area $\langle a\rangle$ [Fig. 1(a)] when the concentration c is increased up to $c\approx 10\%$. We assume that the parameters c, and E_c^B increase the disorder in the lattice. This has an effect that small avalanches contribute to decreasing the average avalanche area $\langle a\rangle$ as observed in the magnets crackle experiments [29]. The role of disorder was supported by the seismic observation [36] where the redistributing of stress has an effect on the ratio between small and big earthquakes in the favor smaller earthquakes.

The distributions of times between two avalanches follow an exponential dependence. These results agree with the result obtained for the BTW model when the avalanches flow down the slope [33]. For all parameters the distribution have the same form, an exponential dependence, only the characteristic time T_0 is changed (Table I). We therefore assume that the frequency spectrums computed by randomly superimposing avalanches, will have the same $1/f^2$ properties as the BTW model. The frequency spectrum computed from a direct realization of the avalanche signal [32] displays white noise rather than a $1/f^\alpha$ dependence. We therefore have to modify the model to obtain a $1/f$ noise.

We have thus presented a deterministic, *non-Abelian* model that shows a dependence of scaling exponents on model parameters similar to that found in the generalized Zhang model [23]. For the case where we allow a random distribution of E_c^B, our results appears to agree with the conclusions in paper [16] where it was reported that quenched randomness of thresholds has no effect on the SOC.

The discussion about the universality of deterministic and stochastic models is therefore not closed yet [23,28]. We can observe the importance of toppling mechanisms as there is a clear dependence of scaling exponents on the model parameters. To support this hypothesis further we need to perform a more detailed study of the avalanche dynamics.

V. CONCLUSION

We have modified the BTW model using a second-higher threshold randomly distributed in a lattice. The model has thus two distinct thresholds, where the level of the second threshold and concentration of the sites with this level are introduced as parameters in the model. A lattice site with the higher threshold behaves as a defect. The model is classified as *non-Abelian*. Computer simulations show that the avalanche dynamics changes distinctly when the model parameters are changed.

We have observed an avalanche area distribution which cannot be approximated by a simple power-law dependence. The distributions change when the avalanches are smaller then a critical avalanche area, and for certain parameters the curve does not follow a simple power law. For avalanches larger than a critical avalanche area, the power-law approximation is valid, but the scaling exponent is not universal.

We have also analyzed a possible presence of $1/f$ noise, but our model does not show this behavior contrary to the BTW model. Clearly, certain modifications are needed to obtain $1/f$ behavior.

Further needed is to find out how to control real systems in order to produce avalanches with the desired statistical parameters. We have demonstrated a reduction of the average avalanche area when the model parameters are appropriately set, and this type of control does not need feedback. If one has a control with feedback it could be possible to tune the threshold level or concentration of defects to reach a desired system response.

ACKNOWLEDGMENTS

The author thanks A. T. Skjeltorp for the stimulus to write this paper, for helpful remarks, and for kind hospitality at the Institute for Energy Technology at Kjeller, where part of this work was done. We also thank G. Helgesen for his reading of the manuscript. We acknowledge the financial support of the Slovak Ministry of Education Grant: Nor/Slov. We are also thankful for computational support from A. Dirner, J. Ondrej (MOSIX PC cluster at Home University), and J. Astalos (CONDOR PC cluster at Technical University Košice).

[1] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. Lett. **59**, 381 (1987).

[2] P. Bak, C. Tang, and K. Wiesenfeld, Phys. Rev. A **38**, 364 (1988).

[3] H.M. Jaeger, C.-H. Liu, and S.R. Nagel, Phys. Rev. Lett. **62**, 40 (1989).

[4] G.A. Held, D.H. Solina II, D.T. Keane, W.J. Haag, P.M. Horn, and G. Grinstein, Phys. Rev. Lett. **65**, 1120 (1990).

[5] S.R. Nagel, Rev. Mod. Phys. **64**, 321 (1992).

[6] V. Frette, K. Christensen, A. Malthe-Sørenssen, J. Feder, T.

Jøssang, and P. Meakin, Nature (London) **379**, 49 (1996).
[7] K. Christensen, A. Corral, V. Frette, J. Feder, and T. Jøssang, Phys. Rev. Lett. **77**, 107 (1996).
[8] E. Altshuler, O. Ramos, C. Martínez, L.E. Flores, and C. Noda, Phys. Rev. Lett. **86**, 5490 (2001).
[9] K.L. Babcock and R.M. Westervelt, Phys. Rev. Lett. **64**, 2168 (1990).
[10] H.J.S. Feder and J. Feder, Phys. Rev. Lett. **66**, 2669 (1991).
[11] S. Field, J. Witt, F. Nori, and X. Ling, Phys. Rev. Lett. **74**, 1206 (1995).
[12] L.P. Kadanoff, S.R. Nagel, L. Wu, and S.-M. Zhou, Phys. Rev. A **39**, 6524 (1989).
[13] S.S. Manna, J. Phys. A **24**, L363 (1991).
[14] S.S. Manna, Physica A **179**, 249 (1991).
[15] Z. Olami, H.J.S. Feder, and K. Christensen, Phys. Rev. Lett. **68**, 1244 (1992).
[16] I.M. Jánosi and J. Kertész, Physica A **200**, 179 (1993).
[17] K.E. Bassler and M. Paczuski, Phys. Rev. Lett. **81**, 3761 (1998).
[18] E. Milshtein, O. Biham, and S. Solomon, Phys. Rev. E **58**, 303 (1998).
[19] D.A. Head and G.J. Rodgers, e-print cond-mat/9805186v2.
[20] A. Chessa, H.E. Stanley, A. Vespignani, and S. Zapperi, Phys. Rev. E **59**, R12 (1999).
[21] D.V. Ktitarev, S. Lübeck, P. Grassberger, and V.B. Priezzhev, Phys. Rev. E **61**, 81 (2000).
[22] S. Lübeck, Phys. Rev. E **61**, 204 (2000).
[23] O. Biham, E. Milshtein, and O. Malcai, Phys. Rev. E **63**, 061309 (2001).
[24] S. Lise and M. Paczuski, Phys. Rev. E **63**, 036111 (2001).
[25] Y.C. Zhang, Phys. Rev. Lett. **63**, 470 (1989).
[26] D. Dhar, Phys. Rev. Lett. **64**, 1613 (1990).
[27] D. Dhar, Physica A **263**, 4 (1999).
[28] L. Pietronero, A. Vespignani, and S. Zapperi, Phys. Rev. Lett. **72**, 1690 (1994).
[29] J.P. Sethna, K.A. Dahmen, and C.R. Myers, Nature (London) **410**, 242 (2001).
[30] C. Tebaldi, M. De Menech, and A.L. Stella, Phys. Rev. Lett. **83**, 3952 (1999).
[31] S.-D. Zhang, Phys. Lett. A **233**, 317 (1997).
[32] J. Davidsen and H.G. Schuster, Phys. Rev. E **62**, 6111 (2000).
[33] H.J. Jensen, K. Christensen, and H.C. Fogedby, Phys. Rev. B **40**, 7425 (1989).
[34] W.H. Press, B.P. Flannery, S.A. Teukolsky, and W.T. Vetterling, *Numerical Recipes in C* (Cambridge University Press, Cambridge, England, 2001).
[35] S. Lübeck and K.D. Usadel, Phys. Rev. E **55**, 4095 (1997).
[36] M. Wyss and S. Wiemer, Science **290**, 1334 (2000).